

Existence of positive solutions of a superlinear boundary value problem with indefinite weight *

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Abstract

We deal with the existence of positive solutions for a two-point boundary value problem associated with the nonlinear second order equation $u'' + a(x)g(u) = 0$. The weight $a(x)$ is allowed to change its sign. We assume that the function $g: [0, +\infty[\rightarrow \mathbb{R}$ is continuous, $g(0) = 0$ and satisfies suitable growth conditions, so as the case $g(s) = s^p$, with $p > 1$, is covered. In particular we suppose that $g(s)/s$ is large near infinity, but we do not require that $g(s)$ is non-negative in a neighborhood of zero. Using a topological approach based on the Leray-Schauder degree we obtain a result of existence of at least a positive solution that improves previous existence theorems.

1 Introduction

In this paper we are interested in the study of positive solutions for the nonlinear two-point boundary value problem

$$\begin{cases} u'' + a(x)g(u) = 0 \\ u(0) = u(L) = 0, \end{cases} \quad (1.1)$$

where $a: [0, L] \rightarrow \mathbb{R}$ is a Lebesgue integrable function and $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function, where $\mathbb{R}^+ := [0, +\infty[$ denotes the set of non-negative real numbers. We recall that a *positive solution* of (1.1) is an absolutely continuous function $u: [0, L] \rightarrow \mathbb{R}^+$ such that its derivative $u'(x)$ is absolutely continuous, $u(x)$ satisfies (1.1) for a.e. $x \in [0, L]$ and $u(x) > 0$ for every $x \in]0, L[$.

This issue has been considered by many authors. As classical examples, we mention [1, 2, 3, 5, 8, 11] (see also the references therein), where different techniques are used to face this type of problem. Our work benefits from a new approach based on the Leray-Schauder topological degree, so, to obtain a

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positive solution, our goal is to prove that the degree of a suitable operator is non-zero on an open domain of $\mathcal{C}([0, L])$ not containing the trivial solution.

Our assumptions allow the weight function $a(x)$ to change its sign a finite number of times and, concerning the nonlinearity, we suppose that $g(s)$ can change its sign, even an infinite number of times, and that, roughly speaking, it has a superlinear growth at zero and at infinity. More in detail, with respect to the growth of $g(s)/s$ at zero, we assume a very general condition which depends on the sign of $g(s)$ in a right neighborhood of zero.

Our main result states that, under the conditions just presented, problem (1.1) has at least a positive solution. This theorem clearly covers the case $g(s) = s^p$, with $p > 1$. Moreover, the results concerning the BVP (1.1) where is assumed that $a(x)g(s) \geq 0$ for a.e. $x \in [0, L]$ and for all $s \geq 0$ (see [5, 8, 11]) or that $g(s) > 0$ for all $s > 0$, when $a(x)$ is allowed to change sign (see [3, 6, 7]), do not contain our result and, in some cases, are easy consequences of it.

Figure 1 and Figure 2 show examples of nonlinearities $g(s)$ satisfying our assumptions and which are not covered by previous results.

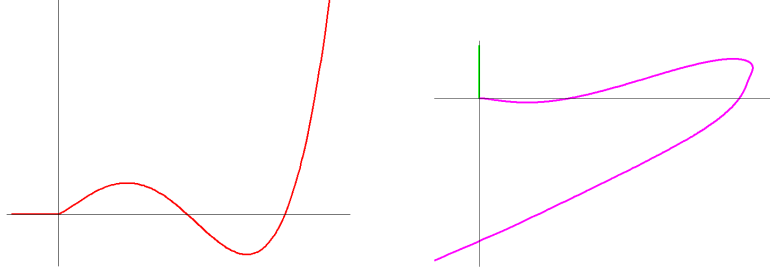


Figure 1: A numerical simulation obtained by setting $I = [0, 1]$, $a(x) = \sin(3\pi x)$ and $g(s) = \min\{20s^{6/5} - 6s^3 + s^4, 400s \arctan(s)\}$. On the left we have shown the graph of $g(s)$. We underline that $g(s)$ changes sign and $g(s)/s \not\rightarrow +\infty$ as $s \rightarrow +\infty$. On the right we have represented the image of the segment $\{0\} \times [0, 12]$ through the Poincaré map in the phase-plane (u, u') . It intersects the negative part of the u' -axis in a point, hence there is a positive initial slope at $x = 0$ from which departs a solution which is positive on $]0, 1[$ and vanishes at $x = 1$.

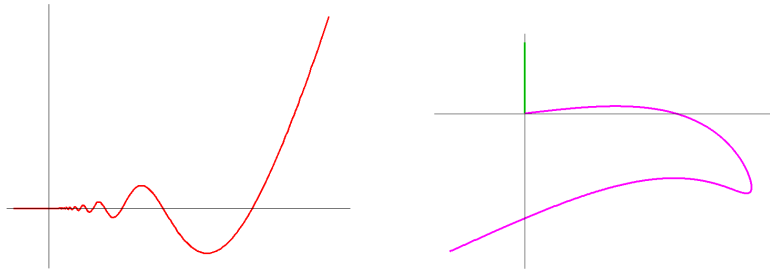


Figure 2: A numerical simulation obtained by setting $I = [0, 1]$, $a(x) = \sin(7\pi x)$ and $g(s) = s^3 + s^2 \sin(1/s)$. On the left we have shown the graph of $g(s)$. The nonlinearity $g(s)$ changes sign an infinite number of times in every neighborhood of zero. On the right we have represented the image of the segment $\{0\} \times [0, 16]$ through the Poincaré map in the phase-plane (u, u') .

The plan of the paper is as follows. In Section 2 we present some basic facts. More in detail we list the hypotheses and we introduce an equivalent fixed point problem that permits to face the problem with a topological approach. In fact, using the technical assumptions, we are able to compute the degree on suitable small and large balls, in the same spirit of [6].

In Section 3 we present our main result. The theorem we state is an immediate corollary of the results exhibited in the previous section. In particular, we prove that the topological degree is non-zero on an annular domain. Therefore a nontrivial fixed point exists, this corresponds to a positive solution (using a standard maximum principle). Straightforward corollaries are then obtained.

Section 4 shows an important existence result of radially symmetric solutions on annular domains.

2 Preliminaries

In this section we state the hypotheses on $a(x)$ and on $g(s)$, we recall some classical results and we prove two preliminary lemmas that are then employed in Section 3 for the main result.

Consider the nontrivial closed interval $[0, L]$, pointing out that different choices of a nontrivial compact interval contained in \mathbb{R} can be made. Let $a: [0, L] \rightarrow \mathbb{R}$ be a L^1 -weight function. Clearly the case of a continuous function can be treated as well. We assume that

(H1) *there exist $m \geq 1$ intervals I_1, \dots, I_m , closed and pairwise disjoint, such that*

$$\begin{aligned} a(x) &\geq 0, \quad \text{for a.e. } x \in \bigcup_{i=1}^m I_i; \\ a(x) &\leq 0, \quad \text{for a.e. } x \in [0, L] \setminus \bigcup_{i=1}^m I_i. \end{aligned}$$

We underline that assumption (H1) trivially includes the case where $a(x) \geq 0$ for a.e. $x \in [0, L]$, taking $m = 1$ and $I_1 = [0, L]$. As standard notation, we define

$$a^+(x) := \max\{a(x), 0\}, \quad a^-(x) := \max\{-a(x), 0\}.$$

Concerning the nonlinearity, we suppose that $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function such that

(H2) $g(0) = 0$ and $g \not\equiv 0$.

We set

$$g_0^{inf} := \liminf_{s \rightarrow 0^+} \frac{g(s)}{s} > -\infty, \quad g_0^{sup} := \limsup_{s \rightarrow 0^+} \frac{g(s)}{s} < +\infty$$

and

$$g_\infty := \liminf_{s \rightarrow +\infty} \frac{g(s)}{s} > 0.$$

We stress that we do not suppose $g(s) \geq 0$ on \mathbb{R}^+ and, in particular, it is not required that $g(s) > 0$ for all $s > 0$ (cf. [5, 6, 7, 8]). Consequently, the

nonlinearity $g(s)$ could be non-negative, non-positive or it could change sign, even an infinite number of times, on a compact neighborhood of zero.

Now we show how the superlinearity of g is expressed at zero and at infinity. As first step we impose a condition on the growth of $g(s)/s$ at 0, depending on the sign of $g(s)$. Precisely we assume that

(H3) • if there exists $\delta > 0$ such that $g(s) \geq 0$, for all $s \in [0, \delta]$, it holds that

$$a^+(x) \not\equiv 0 \text{ on } [0, L] \quad \text{and} \quad g_0^{sup} < \lambda_0^+,$$

where $\lambda_0^+ > 0$ is the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a^+(x) \varphi = 0, \quad \varphi(0) = \varphi(L) = 0;$$

• if there exists $\delta > 0$ such that $g(s) \leq 0$, for all $s \in [0, \delta]$, it holds that

$$a^-(x) \not\equiv 0 \text{ on } [0, L] \quad \text{and} \quad g_0^{inf} > -\lambda_0^-,$$

where $\lambda_0^- > 0$ is the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a^-(x) \varphi = 0, \quad \varphi(0) = \varphi(L) = 0;$$

• if $g(s)$ changes sign an infinite number of times in every neighborhood of zero, it holds that

$$a(x) \not\equiv 0 \text{ on } [0, L] \quad \text{and} \quad -\lambda_0 < g_0^{inf} \leq g_0^{sup} < \lambda_0,$$

where $\lambda_0 > 0$ is the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda |a(x)| \varphi = 0, \quad \varphi(0) = \varphi(L) = 0.$$

The functions $a(x)$ and $g(s)$ introduced in Figure 1 satisfy the first condition of hypothesis (H3), while the example shown in Figure 2 corresponds to the third case.

As second step we define the superlinear behavior at infinity. We suppose that

(H4) for all $i \in \{1, \dots, m\}$

$$a(x) \not\equiv 0 \text{ on } I_i \quad \text{and} \quad g_\infty > \lambda_1^i,$$

where $\lambda_1^i > 0$ is the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda a^+(x) \varphi = 0, \quad \varphi|_{\partial I_i} = 0.$$

Now we describe the topological approach we adopt to face problem (1.1). Our first goal is to introduce a completely continuous operator and to define an equivalent fixed point problem.

Let $\tilde{g}: \mathbb{R} \rightarrow \mathbb{R}$ be the standard extension of $g(s)$ defined as

$$\tilde{g}(s) = \begin{cases} g(s), & \text{if } s \geq 0; \\ 0, & \text{if } s \leq 0. \end{cases}$$

We deal with the boundary value problem

$$\begin{cases} u'' + a(x)\tilde{g}(u) = 0 \\ u(0) = u(L) = 0. \end{cases} \quad (2.1)$$

From conditions (H2) and (H3) and by a classical maximum principle (cf. [6, 9]), it follows that all possible solutions of (2.1) are non-negative. Moreover, if these solutions are nontrivial, then they are strictly positive on $]0, L[$ and hence positive solutions of (1.1).

The next step is to define the classical operator $\Phi: \mathcal{C}([0, L]) \rightarrow \mathcal{C}([0, L])$ by

$$(\Phi u)(x) := \int_0^L G(x, \xi) a(\xi) \tilde{g}(u(\xi)) d\xi, \quad (2.2)$$

where $G(x, s)$ is the Green function associated to the equation $u'' + u = 0$ with the two-point boundary condition. The operator Φ is completely continuous in $\mathcal{C}([0, L])$, endowed with the sup-norm $\|\cdot\|_\infty$, and such that u is a fixed point of Φ if and only if u is a solution of (2.1). Therefore we have transformed problem (1.1) into an equivalent fixed point problem.

We close this section by proving two technical lemmas that allow us to find a nontrivial fixed point of Φ , hence a positive solution of (1.1). The approach we use now is based on the Leray-Schauder topological degree and it is in the same spirit of [6].

Using this first lemma we are able to compute the degree of $Id - \Phi$ on small balls.

Lemma 2.1. *There exists $r_0 > 0$ such that*

$$\deg(Id - \Phi, B(0, r), 0) = 1, \quad \forall 0 < r \leq r_0.$$

Proof. We divide the proof in two steps.

Step 1. We prove that there exists $r_0 > 0$ such that every solution $u(x) \geq 0$ of the two-point BVP

$$\begin{cases} u'' + \vartheta a(x)g(u) = 0, & 0 \leq \vartheta \leq 1, \\ u(0) = u(L) = 0 \end{cases} \quad (2.3)$$

satisfying $\max_{x \in [0, L]} u(x) \leq r_0$ is such that $u(x) = 0$, for all $x \in [0, L]$.

The proof of this first step is given only when there exists $\delta > 0$ such that $g(s) \geq 0$, for all $s \in [0, \delta]$. The two remaining cases can be treated in an analogous way.

Using condition (H3), we fix $0 < r_0 < \delta$ such that

$$\frac{g(s)}{s} < \lambda_0^+, \quad \forall 0 < s \leq r_0.$$

Now, suppose by contradiction that there exist $\vartheta \in [0, 1]$ and a positive solution $u(x) \not\equiv 0$ of (2.3) such that $\max_{x \in [0, L]} u(x) = r$ for some $0 < r \leq r_0$. The choice of r_0 and the maximum principle imply that

$$0 \leq \vartheta g(u(x)) < \lambda_0^+ u(x), \quad \text{for all } x \in]0, L[.$$

Let φ be a positive eigenfunction of

$$\begin{cases} \varphi'' + \lambda_0^+ a^+(x)\varphi = 0 \\ \varphi(0) = \varphi(L) = 0. \end{cases}$$

We stress that $\varphi(x) > 0$, for all $x \in]0, L[$. Using a Sturm comparison argument, we attain

$$\begin{aligned} 0 &= [u'(x)\varphi(x) - u(x)\varphi'(x)]_{x=0}^{x=L} \\ &= \int_0^L \frac{d}{dx} [u'(x)\varphi(x) - u(x)\varphi'(x)] dx \\ &= \int_0^L [u''(x)\varphi(x) - u(x)\varphi''(x)] dx \\ &= \int_0^L [-\vartheta a(x)g(u(x))\varphi(x) + u(x)\lambda_0^+ a^+(x)\varphi(x)] dx \\ &\geq \int_0^L [\lambda_0^+ u(x) - \vartheta g(u(x))] a^+(x)\varphi(x) dx \\ &> 0, \end{aligned}$$

a contradiction.

Step 2. Computation of the degree. Let us fix $0 \leq \vartheta \leq 1$. As remarked when we have introduced the operator Φ , the maximum principle ensures that every fixed point in $\mathcal{C}([0, L])$ of the operator $\vartheta\Phi$ is non-negative and, moreover, $u \in \mathcal{C}([0, L])$ satisfies $u = \vartheta\Phi(u)$ if and only if u is a solution of the equation (2.3). Therefore, setting $r \in]0, r_0]$, *Step 1* implies that $\|u\|_\infty \neq r$ and hence

$$u \neq \vartheta\Phi(u), \quad \forall \vartheta \in [0, 1], \quad \forall u \in \partial B(0, r).$$

By the homotopic invariance property of the topological degree, we obtain that

$$\deg(Id - \Phi, B(0, r), 0) = \deg(Id, B(0, r), 0) = 1.$$

□

Now we compute the degree on large balls.

Lemma 2.2. *There exists $R^* > 0$ such that*

$$\deg(Id - \Phi, B(0, R), 0) = 0, \quad \forall R \geq R^*.$$

Proof. We divide the proof in two steps.

Step 1. A priori bounds for u on each I_i . For each $i \in \{1, \dots, m\}$, we prove that there exists $R_i > 0$ such that for each L^1 -Carathéodory function $h: [0, L] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ with

$$h(x, s) \geq a(x)g(s), \quad \text{a.e. } x \in I_i, \quad \forall s \geq 0,$$

every solution $u(x) \geq 0$ of the two-point BVP

$$\begin{cases} u'' + h(x, u) = 0 \\ u(0) = u(L) = 0 \end{cases} \quad (2.4)$$

satisfies $\max_{x \in I_i} u(x) < R_i$.

We fix an index $i \in \{1, \dots, m\}$ and set $I_i := [\sigma_i, \tau_i]$. Let $0 < \varepsilon < (\tau_i - \sigma_i)/2$ be fixed such that

$$a^+(x) \not\equiv 0 \quad \text{on } I_i^\varepsilon,$$

where $I_i^\varepsilon := [\sigma_i + \varepsilon, \tau_i - \varepsilon]$, and such that the first positive eigenvalue $\hat{\lambda}$ of the eigenvalue problem

$$\begin{cases} \varphi'' + \lambda a^+(x) \varphi = 0 \\ \varphi|_{\partial I_i^\varepsilon} = 0 \end{cases} \quad (2.5)$$

is such that

$$0 < \hat{\lambda} < g_\infty.$$

The existence of ε is ensured by the continuity of the eigenvalue as function of the boundary condition (see [4, 12]) and by hypothesis (H4). From the previous inequality it follows that there exists a constant $\tilde{R} > 0$ such that

$$g(s) > \hat{\lambda}s, \quad \forall s \geq \tilde{R}.$$

By contradiction, suppose there is not a constant $R_i > 0$ with the properties listed above. So, for each integer $n > 0$ there exists a solution $u_n \geq 0$ of (2.4) with $\max_{x \in I_i} u_n(x) =: \hat{R}_n > n$.

We claim that there exists an integer $N \geq \tilde{R}$ such that $u_n(x) > \tilde{R}$ for every $x \in I_i^\varepsilon$ and $n \geq N$. If it is not true, for every integer $n \geq \tilde{R}$ there is an integer $\hat{n} \geq n$ and $x_{\hat{n}} \in I_i^\varepsilon$ such that $u_{\hat{n}}(x_{\hat{n}}) = \tilde{R}$. We note that the solution $u_{\hat{n}}(x)$ is concave on each subinterval of I_i where $u_{\hat{n}}(x) \geq \tilde{R}$, since $a(x)g(s) \geq 0$ for a.e. $x \in I_i$ and for all $s \geq \tilde{R}$. Then, without loss of generality, we can assume that there exists a maximum point $\hat{x}_{\hat{n}} \in I_i$ of $u_{\hat{n}}$ such that $u_{\hat{n}}(x) > \tilde{R}$ for all x between $x_{\hat{n}}$ and $\hat{x}_{\hat{n}}$ (if necessary, we change the choice of $x_{\hat{n}}$). From the assumptions, it follows that

$$\hat{n} < \hat{R}_{\hat{n}} = u_{\hat{n}}(\hat{x}_{\hat{n}}) = u_{\hat{n}}(x_{\hat{n}}) + \int_{x_{\hat{n}}}^{\hat{x}_{\hat{n}}} u'_{\hat{n}}(\xi) d\xi \leq \tilde{R} + (\tau_i - \sigma_i)|u'_{\hat{n}}(x_{\hat{n}})|. \quad (2.6)$$

Since $h(x, s)$ is a L^1 -Carathéodory function, there exists $\gamma_{\tilde{R}} \in L^1([0, L], \mathbb{R}^+)$ such that $|h(x, s)| \leq \gamma_{\tilde{R}}(x)$, for a.e. $x \in [0, L]$ and for all $|s| \leq \tilde{R}$. Then, we fix a constant $C > 0$ such that

$$C > \frac{\tilde{R}}{\varepsilon} + \|\gamma_{\tilde{R}}\|_{L^1}.$$

Using (2.6), we have that for every $n \geq (\tau_i - \sigma_i)C + \tilde{R}$ there exists $\hat{n} \geq n$ and $x_{\hat{n}} \in I_i^\varepsilon$ such that $u_{\hat{n}}(x_{\hat{n}}) = \tilde{R}$ and $|u'_{\hat{n}}(x_{\hat{n}})| > C$. Let us fix $n \geq (\tau_i - \sigma_i)C + \tilde{R}$, $\hat{n} \geq n$ and $x_{\hat{n}} \in I_i^\varepsilon$ with the properties just listed. Suppose that $u'_{\hat{n}}(x_{\hat{n}}) > C$ and consider the interval $[\sigma_i, x_{\hat{n}}]$. If $u'_{\hat{n}}(x_{\hat{n}}) < -C$ we proceed similarly dealing with the interval $[x_{\hat{n}}, \tau_i]$. For every $x \in [\sigma_i, x_{\hat{n}}]$

$$u'_{\hat{n}}(x) = u'_{\hat{n}}(x_{\hat{n}}) + \int_{x_{\hat{n}}}^x u''_{\hat{n}}(\xi) d\xi,$$

then

$$u'_{\hat{n}}(x) > C - \int_x^{x_{\hat{n}}} |h(\xi, u_{\hat{n}}(\xi))| d\xi.$$

From this inequality we obtain that $u_{\hat{n}}(x) \leq \tilde{R}$, for all $x \in [\sigma_i, x_{\hat{n}}]$, and therefore

$$u'_{\hat{n}}(x) > \frac{\tilde{R}}{\varepsilon}, \quad \text{for all } x \in [\sigma_i, x_{\hat{n}}].$$

Then, we obtain

$$\tilde{R} \leq \frac{\tilde{R}}{\varepsilon}(x_{\hat{n}} - \sigma_i) < \int_{\sigma_i}^{x_{\hat{n}}} u'_{\hat{n}}(\xi) d\xi = u_{\hat{n}}(x_{\hat{n}}) - u_{\hat{n}}(\sigma_i) \leq u_{\hat{n}}(x_{\hat{n}}) = \tilde{R},$$

a contradiction. Hence the claim is proved. So, we can fix an integer $N \geq \tilde{R}$ such that $u_n(x) > \tilde{R}$ for every $x \in I_i^\varepsilon$ and for $n \geq N$.

We denote by φ the positive eigenfunction of the eigenvalue problem (2.5) with $\|\varphi\|_\infty = 1$. Then $\varphi(x) > 0$, for every $x \in]\sigma_i + \varepsilon, \tau_i - \varepsilon[$, and $\varphi'(\sigma_i + \varepsilon) > 0 > \varphi'(\tau_i - \varepsilon)$. We remark that $u_n(\sigma_i + \varepsilon) > 0$ and $u_n(\tau_i - \varepsilon) > 0$, for every integer n , employing the maximum principle.

Using a Sturm comparison argument, for each $n \geq N$, we obtain

$$\begin{aligned} 0 &> u_n(\tau_i - \varepsilon)\varphi'(\tau_i - \varepsilon) - u_n(\sigma_i + \varepsilon)\varphi'(\sigma_i + \varepsilon) \\ &= \left[u_n(x)\varphi'(x) - u'_n(x)\varphi(x) \right]_{x=\sigma_i+\varepsilon}^{x=\tau_i-\varepsilon} \\ &= \int_{\sigma_i+\varepsilon}^{\tau_i-\varepsilon} \frac{d}{dx} \left[u_n(x)\varphi'(x) - u'_n(x)\varphi(x) \right] dx \\ &= \int_{I_i^\varepsilon} \left[u_n(x)\varphi''(x) - u''_n(x)\varphi(x) \right] dx \\ &= \int_{I_i^\varepsilon} \left[-u_n(x)\hat{\lambda}a^+(x)\varphi(x) + h(x, u_n(x))\varphi(x) \right] dx \\ &= \int_{I_i^\varepsilon} \left[h(x, u_n(x)) - \hat{\lambda}a^+(x)u_n(x) \right] \varphi(x) dx \\ &\geq \int_{I_i^\varepsilon} \left[a(x)g(u_n(x)) - \hat{\lambda}a^+(x)u_n(x) \right] \varphi(x) dx \\ &= \int_{I_i^\varepsilon} \left[g(u_n(x)) - \hat{\lambda}u_n(x) \right] a^+(x)\varphi(x) dx \\ &\geq 0, \end{aligned}$$

a contradiction.

Step 2. Computation of the degree. We stress that the constant R_i , $i \in \{1, \dots, m\}$, does not depend on the function $h(x, s)$. Define

$$R^* := \max_{i=1, \dots, m} R_i + \tilde{R} > 0$$

and fix a radius $R \geq R^*$.

We denote by $\mathbb{1}_A$ the characteristic function of the set $A := \bigcup_{i=1}^m I_i$. Let us define $v(x) := \int_I G(x, s) \mathbb{1}_A(s) ds$. Using a classical result (see [4, Theorem 3.1] or [10, Lemma 1.1]), if we show that

$$u \neq \Phi(u) + \alpha v, \quad \text{for all } u \in \partial B(0, R) \text{ and } \alpha \geq 0, \quad (2.7)$$

the theorem is proved.

Let $\alpha \geq 0$. The maximum principle ensures that any nontrivial solution $u \in \mathcal{C}([0, L])$ of $u = \Phi(u) + \alpha v$ is a non-negative solution of $u'' + a(x)\tilde{g}(u) + \alpha \mathbb{1}_A(x) = 0$ with $u(0) = u(L) = 0$. Hence, u is a non-negative solution of (2.4) with

$$h(x, s) = a(x)g(s) + \alpha \mathbb{1}_A(x).$$

By definition, we have that $h(x, s) \geq a(x)g(s)$, for a.e. $x \in A$ and for all $s \geq 0$, and $h(x, s) = a(x)g(s)$, for a.e. $x \in [0, L] \setminus A$ and for all $s \geq 0$. By the convexity of the solution u on the intervals of $[0, L] \setminus A$ where $u(x) \geq \tilde{R}$, we obtain that

$$\|u\|_\infty = \max_{x \in [0, L]} u(x) \leq \max \left\{ \max_{x \in A} u(x), \tilde{R} \right\}.$$

From *Step 1* and the definition of \tilde{R} we deduce that $\|u\|_\infty < R^* \leq R$. Then (2.7) is proved and the theorem follows. \square

3 The main result

In this section we apply the two technical lemmas just proved to obtain the existence of a positive solution to the two-point boundary value problem (1.1). More in detail, we use the additivity of the topological degree to provide the existence of a nontrivial fixed point of the operator Φ defined in (2.2).

A first immediate consequence of Lemma 2.1 and Lemma 2.2 is our main theorem.

Theorem 3.1. *Let $a: [0, L] \rightarrow \mathbb{R}$ be a L^1 -function and $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function satisfying (H1), (H2), (H3) and (H4). Then there exists at least a positive solution of the two-point boundary value problem (1.1).*

Proof. Let r_0 be as in Lemma 2.1 and R^* be as in Lemma 2.2. We observe that $0 < r_0 < R^* < +\infty$. From the additivity property and the two preliminary lemmas it follows that

$$\begin{aligned} \deg(Id - \Phi, B(0, R^*) \setminus B[0, r_0], 0) &= \\ &= \deg(Id - \Phi, B(0, R^*), 0) - \deg(Id - \Phi, B(0, r_0), 0) = \\ &= 0 - 1 = -1 \neq 0. \end{aligned}$$

Then there exists a nontrivial fixed point of Φ and hence a corresponding positive solution of (1.1), as already remarked. \square

From Theorem 3.1 we easily achieve the following two results.

Corollary 3.1. *Let $a: [0, L] \rightarrow \mathbb{R}$ be a L^1 -function and $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function satisfying (H1) and (H2). Assume that*

$$g'(0) = \lim_{s \rightarrow 0^+} \frac{g(s)}{s} = 0,$$

and, for each $i \in \{1, \dots, m\}$, suppose that $a(x) \not\equiv 0$ on I_i and

$$g'(\infty) := \lim_{s \rightarrow +\infty} \frac{g(s)}{s} = +\infty.$$

Then there exists at least a positive solution of the two-point BVP (1.1).

Corollary 3.2. *Let $a: [0, L] \rightarrow \mathbb{R}$ be a L^1 -function satisfying (H1) and such that $a(x) \not\equiv 0$ on I_i , for each $i \in \{1, \dots, m\}$. Let $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function satisfying (H2) and such that $g'(0) = 0$ and $g'(\infty) = \Lambda > 0$. Then there exists $\lambda^* > 0$ such that, for each $\lambda > \lambda^*$, the two-point BVP*

$$\begin{cases} u'' + \lambda a(x)g(u) = 0 \\ u(0) = u(L) = 0 \end{cases}$$

has at least a positive solution.

Although hypothesis (H1) is more interesting when the set $[0, L] \setminus \bigcup_{i=1}^m I_i$ is not negligible, we can consider a weight $a(x) \geq 0$ for a.e. $x \in [0, L]$, as previously observed. In that situation Corollary 3.1 ensures the existence of a positive solution in the superlinear case (i.e. $g'(0) = 0$ and $g'(\infty) = +\infty$), provided that $a \not\equiv 0$. No sign condition on the function $g(s)$ is required. Thus we have extended [5, Theorem 1], attained as an application of Krasnosel'skiĭ fixed point Theorem.

Remark 3.1. Our approach is based on the definition of a fixed point problem which is equivalent to the boundary value problem considered. It is clear that we could deal with different conditions at the boundary of $[0, L]$ like $u'(0) = u(L) = 0$ or $u(0) = u'(L) = 0$, since a suitable maximum principle and a Green function (cf. [5]) are available to define an equivalent fixed point problem and to adapt the scheme shown in this paper.

4 Radially symmetric solutions

We denote by $\|\cdot\|$ the Euclidean norm in \mathbb{R}^N (for $N \geq 2$). Let

$$\Omega := B(0, R_2) \setminus B[0, R_1] = \{x \in \mathbb{R}^N : R_1 < \|x\| < R_2\}$$

be an open annular domain, with $0 < R_1 < R_2$. Let $a: [R_1, R_2] \rightarrow \mathbb{R}$ be a continuous function. In this section we consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta u = a(\|x\|)g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (4.1)$$

and we are interested in the existence of positive solutions of (4.1), namely classical solutions such that $u(x) > 0$ for all $x \in \Omega$.

Since we look for radially symmetric solutions of (4.1), our study can be reduced to the search of positive solutions of the two-point boundary value problem

$$w''(r) + \frac{N-1}{r}w'(r) + a(r)g(w(r)) = 0, \quad w(R_1) = w(R_2) = 0. \quad (4.2)$$

Indeed, if $w(r)$ is a solution of (4.2), then $u(x) := w(\|x\|)$ is a solution of (4.1). Using the standard change of variable

$$t = h(r) := \int_{R_1}^r \xi^{1-N} d\xi$$

and defining

$$L := \int_{R_1}^{R_2} \xi^{1-N} d\xi, \quad r(t) := h^{-1}(t) \quad \text{and} \quad v(t) = w(r(t)),$$

we transform (4.2) into the equivalent problem

$$v''(t) + r(t)^{2(N-1)}a(r(t))g(v(t)) = 0, \quad v(0) = v(L) = 0. \quad (4.3)$$

Consequently, the two-point boundary value problem (4.3) is of the same form of (1.1) considering $r(t)^{2(N-1)}a(r(t))$ as weight function.

Clearly the following result holds.

Theorem 4.1. *Let $a: [R_1, R_2] \rightarrow \mathbb{R}$ and $g: \mathbb{R}^+ \rightarrow \mathbb{R}$ be continuous functions satisfying (H1), (H2), (H3) and (H4). Then problem (4.1) has at least a positive solution.*

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